

# Magnetic fields and chiral asymmetry in the early hot universe

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**Abstract.** In this paper, we study analytically the process of external generation and subsequent free evolution of the lepton chiral asymmetry and helical magnetic fields in the early hot universe. This process is known to be affected by the Abelian anomaly of the electroweak gauge interactions. As a consequence, chiral asymmetry in the fermion distribution generates magnetic fields of non-zero helicity, and vice versa. We take into account the presence of thermal bath, which serves as a seed for the development of instability in magnetic field in the presence of externally generated lepton chiral asymmetry. The developed helical magnetic field and lepton chiral asymmetry support each other, considerably prolonging their mutual existence, in the process of ‘inverse cascade’ transferring magnetic-field power from small to large spatial scales. For cosmologically interesting initial conditions, the chiral asymmetry and the energy density of helical magnetic field are shown to evolve by scaling laws, effectively depending on a single combined variable. In this case, the late-time asymptotics of the conformal chiral chemical potential reproduces the universal scaling law previously found in the literature for the system under consideration. This regime is terminated at lower temperatures because of scattering of electrons with chirality change, which exponentially washes out chiral asymmetry. We derive an expression for the termination temperature as a function of the chiral asymmetry and energy density of helical magnetic field.

**Keywords:** primordial magnetic fields, leptogenesis

**ArXiv ePrint:** [1607.04845](https://arxiv.org/abs/1607.04845)

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## 1 Introduction

Observations have established the presence of magnetic field of various magnitudes and on various spatial scales in our universe. Galaxies such as Milky Way contain regular magnetic fields of the order of  $\mu\text{G}$ , while coherent fields of the order of  $100\ \mu\text{G}$  are detected in distant galaxies [1, 2]. There is a strong evidence for the presence of magnetic field in intergalactic medium, including voids [3–6], with strengths exceeding  $\sim 10^{-15}\ \text{G}$ . This supports the idea of cosmological origin of magnetic fields, which are subsequently amplified in galaxies, probably by the dynamo mechanism (see reviews [7–10]).

The origin of cosmological magnetic field is a problem yet to be solved, with several possible mechanisms under discussion. These can broadly be classified into inflationary and post-inflationary scenarios. Both types still face problems to overcome: inflationary magnetic fields are constrained to be rather weak, while those produced after inflation typically have too small coherence lengths (see [7–10] for a review of these mechanisms and assessment of these difficulties). It should also be noted that generation of helical hypermagnetic field prior to the electroweak phase transition may explain the observed baryon asymmetry of the universe [11, 12].

One of the mechanisms of generation of cosmological magnetic fields which is currently under scrutiny is based on the Abelian anomaly of the electroweak interactions [13–15]. If the difference between the number densities of right-handed and left-handed charged fermions in the early hot universe happens to be non-zero (as in the leptogenesis scenario involving physics beyond the standard model; see [16, 17] for reviews), then a specific instability arises with respect to generation of helical (hypercharge) magnetic field. The generated helical magnetic field, in turn, is capable of supporting the fermion chiral asymmetry, thus prolonging its own existence to cosmological temperatures as low as tens of MeV [18]. In this process, magnetic-field power is permanently transferred from small to large spatial scales (the phenomenon known as ‘inverse cascade’). Further investigation of the general properties of the regime of inverse cascade revealed certain universal scaling laws in its late-time asymptotics [19–21].

In this paper, we study analytically the process of generation of helical magnetic field in the early hot universe by an unspecified external source of lepton chiral asymmetry. Helical

magnetic field is produced due to the presence of thermal background, which we extrapolate to all spatial scales, including the super-horizon scales.<sup>1</sup> We consider a simple model of generation of magnetic field which assumes that the source of chiral anomaly maintains a constant value of the (conformal) chiral chemical potential of charged leptons. After generation of magnetic field of near maximal helicity, its evolution is traced in the absence of the external source of lepton chiral asymmetry. In this case, the helical magnetic field and the lepton chiral asymmetry are mutually sustained (decaying slowly) by quantum anomaly until temperatures of the order of tens of MeV, with magnetic-field power being permanently transferred from small to large spatial scales in the regime of inverse cascade. We obtain analytic expressions describing the evolution of the lepton chiral chemical potential and magnetic-field energy density. The evolution of both these quantities exhibits certain scaling behavior, effectively depending on a single combined variable. In this case, the late-time asymptotics of the chiral chemical potential reproduces the universal scaling law previously found in the literature for the system under investigation [19–21]. As the temperature drops down because of the cosmological expansion, the processes of lepton scattering with the change of chirality (the so-called chirality-flipping processes) start playing important role, eventually leading to a rapid decay of the lepton chiral asymmetry. We give an analytic expression for the temperature at which this happens, depending on the initially generated values of the magnetic-field energy density and lepton chiral asymmetry.

## 2 Helical magnetic fields

A spatially flat expanding universe filled by relativistic matter is conveniently described in the comoving conformal coordinate system  $(\eta, \mathbf{x})$  with the conformal time  $\eta$  and scale factor  $a(\eta)$  entering the metric line element as  $ds^2 = a^2(\eta) (d\eta^2 - d\mathbf{x}^2)$ . By rescaling the conformal coordinates  $(\eta, \mathbf{x})$ , one can suitably normalize the scale factor  $a(\eta)$ .

A divergence-free statistically homogeneous and isotropic cosmological magnetic field has the following general Fourier representation of the two-point correlation function<sup>2</sup> [22]:

$$\langle B_i(\mathbf{k}) B_j^*(\mathbf{k}') \rangle = (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') \left[ P_{ij}(\mathbf{k}) S(k) + i \epsilon_{ijs} \hat{k}_s A(k) \right], \quad (2.1)$$

where  $\hat{k}_i = k_i/k$ ,  $P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$  is the symmetric projector to the plane orthogonal to  $\mathbf{k}$ , and  $\epsilon_{ijk}$  is the normalized totally antisymmetric tensor.

It is useful to introduce the helicity components  $B_{\pm}(\mathbf{k})$  of the magnetic field via

$$B_i(\mathbf{k}) = B_+(\mathbf{k}) e_i^+(\mathbf{k}) + B_-(\mathbf{k}) e_i^-(\mathbf{k}), \quad (2.2)$$

where the complex basis  $e_i^{\pm} = \frac{1}{\sqrt{2}} (e_i^1 \pm i e_i^2)$  is formed from a right-handed (with respect to the orientation  $\epsilon_{ijk}$ ) and orthonormal (with respect to the metric  $\delta_{ij}$ ) basis  $\mathbf{e}^1(\mathbf{k})$ ,  $\mathbf{e}^2(\mathbf{k})$ ,  $\mathbf{e}^3(\mathbf{k}) = \mathbf{k}/k$ . The coefficients of the symmetric and antisymmetric parts of the correlation function are then expressed through these components as follows:

$$\langle B_-(\mathbf{k}) B_-^*(\mathbf{k}') + B_+(\mathbf{k}) B_+^*(\mathbf{k}') \rangle = 2 (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') S(k), \quad (2.3)$$

$$\langle B_-(\mathbf{k}) B_-^*(\mathbf{k}') - B_+(\mathbf{k}) B_+^*(\mathbf{k}') \rangle = 2 (2\pi)^3 \delta(\mathbf{k} - \mathbf{k}') A(k). \quad (2.4)$$

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<sup>1</sup>The spectral properties of magnetic fields on superhorizon spatial scales depend on a concrete model of generation of primordial magnetic fields (see [8–10] for recent reviews).

<sup>2</sup>The quantities  $B_i$  are the components of the so-called comoving magnetic field, which is related to the observable magnetic field strength  $\mathbf{B}_{\text{obs}}$  by the equation  $\mathbf{B} = a^2 \mathbf{B}_{\text{obs}}$ . The spatial vector indices are treated by using the Kronecker delta-symbol, and their position does not matter.

We note an obvious constraint  $|A(k)| \leq S(k)$ .

The spectrum  $A(k)$  of the magnetic-field correlation function characterizes the difference in the power between the left-handed and right-handed magnetic field, i.e., its helicity. The spectrum  $S(k)$  characterizes the magnetic field energy density. In the case of so-called maximally helical magnetic field, one has  $|A(k)| = S(k)$ , and magnetic field is dominated by its left-handed or right-handed part, depending on the sign of  $A(k)$ .

In this paper, we consider the effects of Abelian anomaly in the presence of spatially homogeneous chiral asymmetry.<sup>3</sup> In this case, the evolution of the comoving magnetic field in conformal coordinates in cosmic plasma with high conductivity  $\sigma$  takes the form

$$\frac{\partial \mathbf{B}}{\partial \eta} = \frac{1}{\sigma_c} \nabla^2 \mathbf{B} - \frac{\alpha \Delta \mu}{\pi \sigma_c} \text{rot } \mathbf{B}, \quad (2.5)$$

where  $\Delta \mu \equiv a(\mu_L - \mu_R)$  is the spatially homogeneous difference between the (conformal) chemical potentials of the left-handed and right-handed charged leptons,  $\sigma_c \equiv a\sigma \approx \text{const}$  [25, 26] characterizes the plasma conductivity, and  $\alpha \approx 1/137$  is the fine structure constant. The last term in equation (2.5) is connected with the anomalous current in Maxwell's equations [13–15, 27–30].

Using equations (2.1) and (2.5), one can obtain the following system of equations for the spectra  $S(k, \eta)$  and  $A(k, \eta)$  (see [18]):

$$\frac{\partial S}{\partial \eta} = -\frac{2k^2}{\sigma_c} (S - S_{\text{eq}}) + \frac{2\alpha k}{\pi \sigma_c} \Delta \mu A, \quad (2.6)$$

$$\frac{\partial A}{\partial \eta} = -\frac{2k^2}{\sigma_c} A + \frac{2\alpha k}{\pi \sigma_c} \Delta \mu S. \quad (2.7)$$

In equation (2.6), we have added a term with the thermal equilibrium distribution

$$S_{\text{eq}}(k, \eta) = \frac{k}{e^{k/aT} - 1}, \quad (2.8)$$

whose role is to ensure relaxation of the spectral energy distribution  $S$  to its equilibrium value  $S_{\text{eq}}$  rather than to zero. This mechanism will not work in the long-wavelength domain  $k \lesssim 1/\eta$ , which is not causally connected in the expanding hot universe. This, however, will be of no practical importance, since the anomalous term in equation (2.6) will dominate in this spectral region. The initial spectra in the domain of small values of  $k$  will also depend on their cosmological origin. We do not consider this issue in the present paper, assuming the initial spectrum to be given by (2.8) on all scales.

In an early radiation-dominated universe expanding adiabatically with the bulk matter in local thermal equilibrium, the entropy density  $(aT)^3 g_*$  remains constant. Here,  $g_*$  is the number of relativistic degrees of freedom  $g_*$  in thermal equilibrium. In the range of temperatures  $80 \text{ GeV} < T < 150 \text{ MeV}$ , the value of  $g_*$  changes insignificantly from about 86 to 72, and at the quantum-chromodynamical crossover, at  $T \simeq 150 \text{ MeV}$ , drops to about 17. The quantity  $g_*^{1/3}$  thus drops from about 4.4 to 2.6, and we can see that the product  $aT$  remains constant to a great extent. It is then convenient to normalize the scale factor as the inverse of the temperature,  $a = 1/T$ . With this choice, we have  $a = \eta/M_*$ , where  $M_* = (45/4\pi^3 g_*)^{1/2} M_{\text{P}} \simeq 10^{18} \text{ GeV}$  is the effective Planck mass, and  $\sigma_c = \sigma/T \approx 70$  is

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<sup>3</sup>Effects of spatial inhomogeneity in chiral relativistic plasma were under investigation in [23, 24].

constant in time [18, 25, 26]. The equilibrium spectrum (2.8) is independent of time and, with this normalization, reads

$$S_{\text{eq}}(k, \eta) \equiv S_0(k) = \frac{k}{e^k - 1}. \quad (2.9)$$

The excess  $\rho_B(\eta)$  of the energy density of magnetic field over the thermal energy density is then determined by

$$\rho_B(\eta) = \frac{T^4}{2\pi^2} \int_0^\infty P(k, \eta) k^2 dk, \quad (2.10)$$

where  $P(k, \eta) \equiv S(k, \eta) - S_0(k)$  is the excess over the thermal power spectrum.

The system of equations (2.6) and (2.7) is supplemented by the evolution equation for the chiral chemical potential [18]:

$$\frac{d\Delta\mu(\eta)}{d\eta} = -\frac{c_\Delta\alpha}{\pi^2} \int_0^\infty \frac{\partial A(k, \eta)}{\partial \eta} k dk - \Gamma_f(\eta)\Delta\mu(\eta) + \beta(\eta). \quad (2.11)$$

Here,  $c_\Delta$  is a numerical constant of order unity (it would be equal to 3/4 in pure quantum electrodynamics) that reflects the particle content of the primordial plasma,  $\beta(\eta)$  is an external source for the evolution of chiral chemical potential which, for definiteness, we assume to be positive, and  $\Gamma_f(\eta)$  is the coefficient of the so-called ‘flipping’ term which describes decay of lepton chiral asymmetry caused by chirality flips in the electroweak interactions.

### 3 Development of helical magnetic fields

Since the conformal conductivity  $\sigma_c$  is constant during the period of evolution under investigation, it is more convenient to work in terms of a rescaled conformal time  $\tau = \eta/\sigma_c$ . With regard of (2.9), system (2.6), (2.7) can then be written in the form

$$\dot{P} = -2k^2 P + \frac{2\alpha k}{\pi} \Delta\mu A, \quad (3.1)$$

$$\dot{A} = -2k^2 A + \frac{2\alpha k}{\pi} \Delta\mu (P + S_0). \quad (3.2)$$

Here and in what follows, the overdot denotes the derivative with respect to  $\tau$ . We then set the initial conditions for system (2.11), (3.1), (3.2) at the formal cosmological singularity  $\tau = 0$  in the form

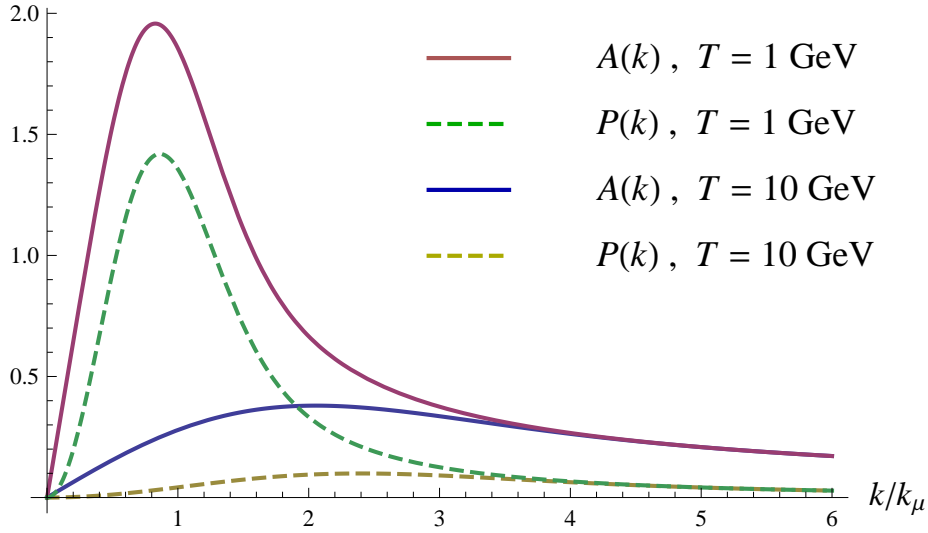
$$P(k, 0) \equiv 0, \quad A(k, 0) \equiv 0, \quad \gamma(0) = 0. \quad (3.3)$$

Hence, we also have  $\Delta\mu(0) = 0$ . Thus, in the presence of lepton chiral asymmetry ( $\Delta\mu \neq 0$ ), generation of the helicity spectrum  $A(k)$  commences, according to (3.2), due to the presence of thermal distribution  $S_0(k)$ .

Solution of system (3.1) and (3.2) with respect to the spectral functions  $P(k, \tau)$  and  $A(k, \tau)$  with the initial conditions (3.3) is given by

$$P(k, \tau) = \frac{2\alpha k}{\pi} S_0(k) e^{-2k^2\tau} \int_0^\tau e^{2k^2\tau'} \sinh(2k [\Psi(\tau) - \Psi(\tau')]) \Delta\mu(\tau') d\tau', \quad (3.4)$$

$$A(k, \tau) = \frac{2\alpha k}{\pi} S_0(k) e^{-2k^2\tau} \int_0^\tau e^{2k^2\tau'} \cosh(2k [\Psi(\tau) - \Psi(\tau')]) \Delta\mu(\tau') d\tau', \quad (3.5)$$



**Figure 1.** Spectrum (3.7) is plotted for  $\Delta\mu = 5 \times 10^{-6}$  ( $k_\mu \simeq 10^{-8}$ ) and at the cosmological plasma temperatures  $T = 10$  GeV (dashed yellow) and  $T = 1$  GeV (dashed green). Spectrum (3.8) is also plotted at  $T = 10$  GeV (solid blue) and  $T = 1$  GeV (solid pink). The spectra increase in magnitude with time. In the region  $k/k_\mu \gg 1$ , the spectra are saturated to their stationary values  $A(k) \simeq S_0(k)k_\mu/k$  and  $P(k) \simeq S_0(k)(k_\mu/k)^2$ .

where

$$\Psi(\tau) = \frac{\alpha}{\pi} \int_0^\tau \Delta\mu(\tau') d\tau'. \quad (3.6)$$

One can see that  $A(k, \tau)$  has the same sign as  $\Delta\mu(\tau)$ , while  $P(k, \tau)$  is always positive.

To estimate the behavior of the spectral functions  $P(k, \tau)$  and  $A(k, \tau)$ , let us evaluate them under the condition  $\Delta\mu \equiv \text{const} > 0$  [which can be ensured by an appropriate behavior of the source  $\beta(\tau)$  in (2.11)]. Using (3.6) and taking the elementary integrals in (3.4) and (3.5), we have

$$P(k, \tau) = \frac{S_0(k)}{(k/k_\mu)^2 - 1} \left( 1 - e^{-2k^2\tau} \left[ \cosh(2k_\mu k\tau) + \frac{k}{k_\mu} \sinh(2k_\mu k\tau) \right] \right), \quad (3.7)$$

$$A(k, \tau) = \frac{S_0(k)}{(k/k_\mu)^2 - 1} \left( \frac{k}{k_\mu} - e^{-2k^2\tau} \left[ \sinh(2k_\mu k\tau) + \frac{k}{k_\mu} \cosh(2k_\mu k\tau) \right] \right), \quad (3.8)$$

where

$$k_\mu = \frac{\alpha\Delta\mu}{\pi}. \quad (3.9)$$

Spectra (3.7) and (3.8) for  $\Delta\mu = 5 \times 10^{-6}$  ( $k_\mu \simeq 10^{-8}$ ) are plotted in figure 1 for temperatures  $T = 10$  GeV and 1 GeV, corresponding to  $\tau \approx 1.2 \times 10^{15}$  and  $1.2 \times 10^{16}$ , respectively.

As can be seen from expressions (3.7) and (3.8) and from figure 1, there arise two characteristic regions of wavenumbers: the region of relatively small  $k$  (of order  $k_\mu$ ), where the spectra keep growing and approach the property  $A(k) \simeq P(k) \gtrsim S_0(k)$  of maximal helicity, and the region of ‘tails’ of these spectra, where they quickly reach the threshold values  $A(k) \simeq S_0(k)k_\mu/k$  and  $P(k) \simeq S_0(k)(k_\mu/k)^2$ .

Indeed, in the region of large wavenumbers  $k \gg k_\mu$ , equation (3.2) is approximated as

$$\dot{A} \approx -2k^2 A + \frac{2\alpha k}{\pi} \Delta\mu S_0 = -2k^2 A + 2k_\mu k S_0, \quad (3.10)$$

with the solution

$$A(k) = S_0(k) \frac{k_\mu}{k} \left(1 - e^{-2k^2\tau}\right), \quad (3.11)$$

that exponentially with time approaches the equilibrium  $A_{\text{eq}} = S_0 k_\mu / k$ . Solution of (3.1) is then given by

$$P(k) = S_0(k) \left(\frac{k_\mu}{k}\right)^2 \left[1 - e^{-2k^2\tau} (1 + 2k^2\tau)\right], \quad (3.12)$$

with a rapid exponential convergence to the equilibrium  $P_{\text{eq}} = S_0 (k_\mu/k)^2$ . For sufficiently slow evolution of  $\Delta\mu(\tau)$ , these expressions for the spectral ‘tails’ will retain their forms, with  $k_\mu(\tau)$  expressed through  $\Delta\mu(\tau)$  by (3.9).

It should be noted that the magnetohydrodynamical description of cosmic plasma cannot be trusted in the domain of large physical wavenumbers  $k/a \gtrsim \alpha T$ , which, in our system of conformal units, corresponds to values of  $k \gtrsim \alpha \sim 10^{-2}$ . This limitation is insignificant for the evolution of magnetic instability developing on much larger spatial scales (as is usually the case in the scenarios under consideration). In the physically relevant domain  $k \lesssim \alpha$ , the thermal spectrum  $S_0(k) \approx 1$ . Therefore, we can replace the factor  $S_0(k)$  by unity in all our equations; however, for the sake of better physical clarity, we will retain it.

In the region of relatively small wavenumbers, the spectra  $P(k, \tau)$  and  $A(k, \tau)$  rapidly become equal to each other, as can be seen from figure 1. In the regime  $P \gg S_0$ , in which the quantity  $S_0$  on the right-hand side of equation (3.2) can be neglected, one can multiply equation (3.1) by  $A(k, \tau)$ , equation (3.2) by  $P(k, \tau)$  and subtract them to obtain an equation relating the spectral functions:

$$(P^2)^\cdot + 4k^2 P^2 = (A^2)^\cdot + 4k^2 A^2. \quad (3.13)$$

This implies the relation

$$P^2(k, \tau) = A^2(k, \tau) + f_0^2(k) e^{-4k^2\tau}, \quad (3.14)$$

where  $f_0^2(k)$  is an integration constant. We see that, if the quantity  $A^2(k, \tau)$  is not decaying or decaying slower than the last exponent in (3.14), then a maximally helical state develops, with  $P \simeq A$ . In this regime, system (3.1), (3.2) reduces to a single equation for  $A(k, \tau) \simeq P(k, \tau)$ :

$$\dot{A} = \left(-2k^2 + \frac{2\alpha k}{\pi} \Delta\mu\right) A. \quad (3.15)$$

#### 4 Evolution of chiral asymmetry and magnetic field

Assuming that a maximally helical configuration quickly develops at some initial time  $\tau_{\text{in}}$ , we are going to establish how it will evolve together with  $\Delta\mu(\tau)$  after the source  $\beta(\tau)$  in (2.11) is switched off.

Let us make the notation  $\Delta\mu_{\text{in}} = \Delta\mu(\tau_{\text{in}})$ ,  $P_{\text{in}}(k) = P(k, \tau_{\text{in}})$ ,  $A_{\text{in}}(k) = A(k, \tau_{\text{in}})$ , and introduce the momentum  $k_{\text{in}}$  similarly to (3.9):

$$k_{\text{in}} = \frac{\alpha \Delta\mu_{\text{in}}}{\pi}. \quad (4.1)$$

The initial spectra can be presented as

$$P_{\text{in}}(k) = P_0 Z\left(\frac{k}{k_{\text{in}}}\right), \quad A_{\text{in}}(k) \approx P_{\text{in}}(k) \left(1 + \frac{k}{k_{\text{in}}}\right), \quad (4.2)$$

where  $Z(x)$  describes the shape of the spectrum, and the factor  $(1 + k/k_{\text{in}})$  is introduced to reflect the relation in the ‘tails’ of the spectra. The normalization constant  $P_0$  is chosen so that

$$\int_0^\infty Z(x) x^2 dx = 1, \quad (4.3)$$

and the initial excess (2.10) of the energy density of magnetic field over the thermal energy density is then equal to

$$\rho_B(\tau_{\text{in}}) = \frac{T_{\text{in}}^4}{2\pi^2} \int_0^\infty P(k, \tau_{\text{in}}) k^2 dk = \frac{T_{\text{in}}^4 k_{\text{in}}^3 P_0}{2\pi^2}. \quad (4.4)$$

It is convenient to relate this quantity to the total radiation energy density by introducing the dimensionless parameter

$$r_B^{\text{in}} \equiv \frac{\rho_B(\tau_{\text{in}})}{\rho_r(\tau_{\text{in}})} = \frac{30\rho_B(\tau_{\text{in}})}{\pi^2 g_* T_{\text{in}}^4} = \frac{15}{2\pi^4 g_*} P_0 k_{\text{in}}^3. \quad (4.5)$$

Asymptotically, as  $\tau \rightarrow \infty$ , the maximum of spectrum (3.7) is reached at  $k = k_\mu/2$ , and one can derive an approximate asymptotic estimate for  $r_B^{\text{in}}$ :

$$r_B^{\text{in}} \simeq \frac{k_{\text{in}}^2}{\pi^4} \sqrt{\frac{1}{2\tau_{\text{in}}}} e^{k_{\text{in}}^2 \tau_{\text{in}}/2}, \quad (4.6)$$

where  $k_{\text{in}}$  is given by (4.1). One can see the exponential dependence of  $r_B^{\text{in}}$  on the (rescaled) conformal time  $\tau_{\text{in}}$  (or temperature  $T_{\text{in}} = M_*/\sigma_c \tau_{\text{in}}$ ) at which the spectrum (3.7) is finally developed by the external source of chiral asymmetry. In what follows, we take  $r_B^{\text{in}}$  and  $\Delta\mu_{\text{in}}$  [related to  $k_{\text{in}}$  by (4.1)] to be our independent parameters.

The subsequent evolution of the spectrum in the domain where  $P(k, \tau) \approx A(k, \tau) \gg S_0(k)$  is described by equation (3.15). Its solution with the initial condition  $A(k, \tau_{\text{in}}) = A_{\text{in}}(k)$  is given by

$$A(k, \tau) = g^2(k, \tau) A_{\text{in}}(k), \quad (4.7)$$

where

$$g(k, \tau) = e^{-k^2 \Delta\tau + k \Delta\Psi(\tau)}, \quad \Delta\tau = \tau - \tau_{\text{in}}, \quad \Delta\Psi(\tau) = \Psi(\tau) - \Psi(\tau_{\text{in}}), \quad (4.8)$$

is the growth factor.

Solution of (2.11) with the zero source  $\beta$  and with the flipping term neglected can be written as

$$\Delta\mu_{\text{in}} - \Delta\mu(\tau) = \frac{c\Delta\alpha}{\pi^2} \int_0^\infty [A(k, \tau) - A_{\text{in}}(k)] k dk. \quad (4.9)$$

The contribution of the ‘tail’ in the distributions  $A(k, \tau)$  and  $A_{\text{in}}(k)$  to the value of the difference  $\Delta\mu_{\text{in}} - \Delta\mu(\tau)$  in this expression is negligibly small. Indeed, the integral over the ‘tail’ region is estimated as

$$\begin{aligned} \frac{c\Delta\alpha}{\pi^2} \int_{k_{\text{tail}}}^\infty [A(k, \tau) - A_{\text{in}}(k)] k dk &\simeq \frac{c\Delta\alpha^2}{\pi^3} [\Delta\mu(\tau) - \Delta\mu_{\text{in}}] \int_{k_{\text{tail}}}^\infty S_0(k) dk \\ &\simeq \frac{c\Delta\alpha^2}{\pi^3} [\Delta\mu(\tau) - \Delta\mu_{\text{in}}], \end{aligned} \quad (4.10)$$



which is much smaller by absolute value than the left-hand side of (4.9) because of the smallness of  $\alpha \approx 1/137$ . Thus, we can ignore the presence of power-law tails in the spectra in (4.9), and write, using (4.2),

$$\Delta\mu_{\text{in}} - \Delta\mu(\tau) = \frac{c\Delta\alpha k_{\text{in}}^2 P_0}{\pi^2} \int_0^\infty [g^2(k_{\text{in}}x, \tau) - 1] Z(x) x dx. \quad (4.11)$$

Dividing this by  $\Delta\mu_{\text{in}}$  and using (4.1) and (4.5), we obtain the estimate

$$1 - \frac{\Delta\mu(\tau)}{\Delta\mu_{\text{in}}} = \frac{\pi c\Delta\alpha^2 g_* r_B^{\text{in}}}{15k_{\text{in}}^2} \int_0^\infty [g^2(k_{\text{in}}x, \tau) - 1] Z(x) x dx. \quad (4.12)$$

For values in a broad typical range of parameters in different cosmological scenarios, the factor in front of the integral in (4.12) is much larger than unity. For instance, for  $g_* = 75$  and  $\Delta\mu_{\text{in}} = 5 \times 10^{-6}$ , this factor is estimated to be  $\sim 10^{13} r_B^{\text{in}}$ , and will be very large for  $r_B^{\text{in}} \gg 10^{-13}$ . Since the left-hand side of (4.12) is always bounded by unity, this implies that the integral on the right-hand side should be extremely small. The relation

$$\int_0^\infty [g^2(k_{\text{in}}x, \tau) - 1] Z(x) x dx \approx 0 \quad (4.13)$$

can then be regarded as an integral equation implicitly expressing the quantity to be found  $\Delta\Psi$  through the known quantity  $\Delta\Phi$  [both enter the function  $g^2(k_{\text{in}}x, \tau)$  under this integral; see (4.8)].

It is convenient to introduce the variables

$$\phi = 2k_{\text{in}}^2 \Delta\tau, \quad \psi = 2k_{\text{in}} \Delta\Psi. \quad (4.14)$$

In terms of these variables, we have

$$g^2(k_{\text{in}}x, \tau) = e^{-\phi x^2 + \psi x}, \quad (4.15)$$

and equation (4.13) establishes the dependence  $\psi(\phi)$ , which is determined only by the form  $Z(x)$  of the initial distribution. To find the dependence  $\psi(\phi)$ , we differentiate (4.13) with respect to  $\phi$ . We obtain the Cauchy problem

$$\frac{d\psi}{d\phi} = \frac{\int_0^\infty e^{-\phi x^2 + \psi x} Z(x) x^3 dx}{\int_0^\infty e^{-\phi x^2 + \psi x} Z(x) x^2 dx}, \quad \psi(0) = 0. \quad (4.16)$$

The evolution of the chiral chemical potential can then be calculated by using (3.6):

$$\frac{\Delta\mu}{\Delta\mu_{\text{in}}} = \frac{\pi}{\alpha \Delta\mu_{\text{in}}} \dot{\Psi} = \psi'(\phi). \quad (4.17)$$

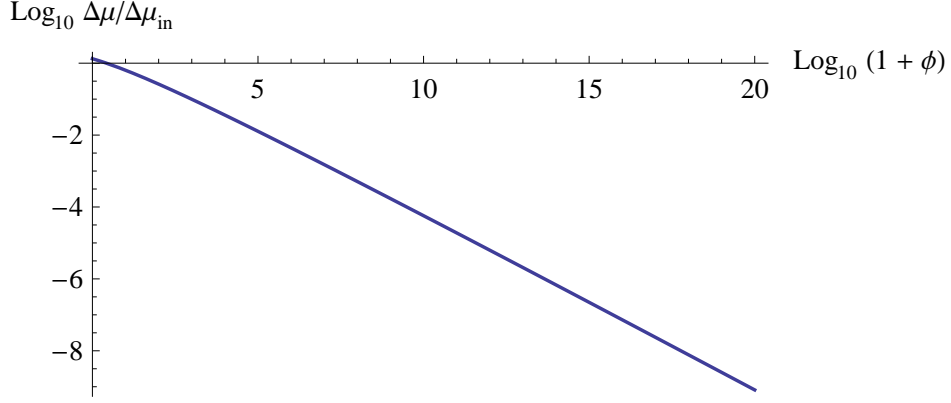
Remarkably, the evolution of the chiral chemical potential  $\Delta\mu$  depends on the initial conditions and on time through a single scaling parameter  $\phi$ , defined in (4.14).

As an example, let us approximate the initial spectrum  $P_{\text{in}}(k)$  with subtracted high-frequency ‘tail’ by a corresponding normalized spectral function with exponential cut-off:<sup>4</sup>

$$Z(x) = 2xe^{-x^2}. \quad (4.18)$$

---

<sup>4</sup>Equation (4.18) gives a correct linear growth at small  $x$ , observed in figure 1.



**Figure 2.** Evolution (4.22) of the chiral chemical potential plotted in logarithmic scale.

Then, introducing the variable

$$\zeta = \frac{\psi}{\sqrt{1+\phi}}, \quad (4.19)$$

we can present problem (4.16) in the form

$$\frac{d\zeta}{d\phi} = \frac{F(\zeta) - \frac{1}{2}\zeta}{1+\phi}, \quad \zeta(0) = 0, \quad (4.20)$$

where

$$F(\zeta) = \frac{\int_0^\infty e^{-x^2+\zeta x} x^4 dx}{\int_0^\infty e^{-x^2+\zeta x} x^3 dx}. \quad (4.21)$$

Equation (4.20) can, in principle, be integrated, and the function  $\zeta(\phi)$  can be found. The evolution (4.17) of the chiral chemical potential is then given by

$$\frac{\Delta\mu}{\Delta\mu_{\text{in}}} = \psi'(\phi) = \frac{F(\zeta(\phi))}{\sqrt{1+\phi}}, \quad (4.22)$$

where  $F(\zeta)$  is given by (4.21). Solution (4.22) is plotted in logarithmic scale in figure 2.

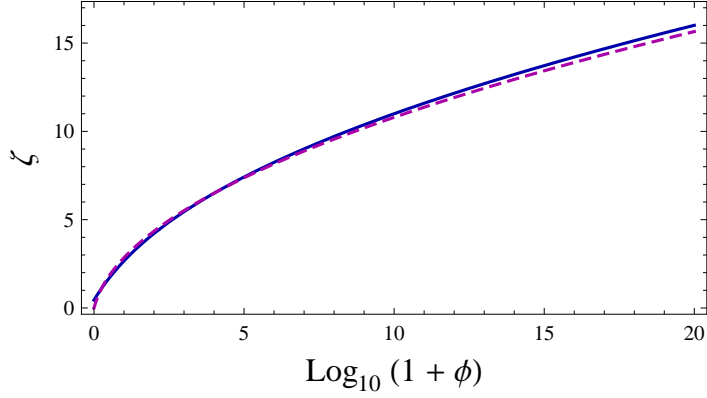
Let us establish the late-time asymptotics of the solution to (4.20), (4.21). In the regime  $\zeta \gg 1$ , we have  $F(\zeta) \approx \zeta/2 + 3/\zeta + \mathcal{O}(\zeta^{-3})$ . Solution of (4.20) in this case behaves as

$$\zeta(\phi) \approx [\text{const} + 6 \log(1+\phi)]^{1/2}, \quad \phi \gg 1. \quad (4.23)$$

This qualitative behavior does not depend on the specific shape (4.18) of the initial spectrum and is caused by the inverse cascade that transfers the spectral power to small-frequency region. Indeed, for large enough values of  $\tau$ , function (4.15) develops a strong Gaussian peak at small values of  $x$ , where  $Z(x)$  behaves rather smoothly (typically, as a power of  $x$ ). Expression (4.16) does not then depend on the concrete form of  $Z(x)$  in this limit.

With the account of asymptotics (4.23), the solution  $\zeta(\phi)$  of the differential equation (4.20), (4.21) can be approximated by the expression

$$\zeta(\phi) = \left[ \sqrt{1 + 6 \log(1+\phi)} - 1 \right]. \quad (4.24)$$



**Figure 3.** Exact solution of system (4.20), (4.21) (solid blue line) versus interpolation (4.24) (dashed pink line).

Numerical integration confirms this approximation within about 1% precision (see figure 3). Then

$$\psi(\phi) = \sqrt{1 + \phi} \zeta(\phi) = \sqrt{1 + \phi} \left[ \sqrt{1 + 6 \log(1 + \phi)} - 1 \right]. \quad (4.25)$$

Using (4.22), we then obtain the universal late-time asymptotics

$$\log \frac{\Delta\mu}{\Delta\mu_{\text{in}}} \approx -\frac{1}{2} \log(1 + \phi) + \frac{1}{2} \log \log(1 + \phi). \quad (4.26)$$

This describes very well the almost ideal power-law behavior observed in figure 2 for large values of  $\phi$ , with the asymptotic power index equal to  $-1/2$ . The late-time asymptotics  $\Delta\mu \propto \eta^{-1/2}$  in the system under consideration has been previously established in [19] (and in [20] with a leading logarithmic correction,  $\Delta\mu \propto \eta^{-1/2} \log^{1/2} \eta$ ), and also noted in [21] in the context of chiral magnetohydrodynamics.

Let us also determine the behavior of the magnetic-field energy density, described by the parameter  $r_B(\tau)$ . For our developed chiral distribution, we have  $P(k, \tau) \approx g^2(k, \tau) P_{\text{in}}(k)$ . Hence,

$$r_B(\tau) = N_B r_B^{\text{in}}, \quad N_B(\phi) = \int_0^\infty Z(x) e^{-\phi x^2 + \psi(\phi)x} x^2 dx. \quad (4.27)$$

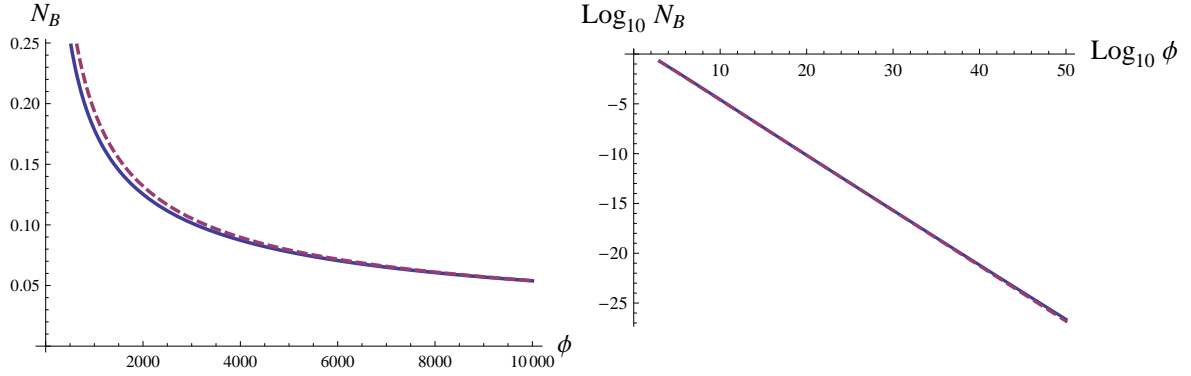
Thus, the energy density of magnetic field also depends on time through a single scaling parameter  $\phi$ , defined in (4.14). Its behavior at large values of  $\phi$  will depend only on the behavior of the function  $Z(x)$  at small  $x$ .

For the initial shape (4.18) of the magnetic-field spectrum, the scaling function  $N_B(\phi)$  is determined by approximations (4.24) and (4.25). Analytic estimate of integral (4.27) gives a rather complicated asymptotics at  $\phi \gg 1$ :

$$N_B(\phi) \propto \frac{\log^{3/2} \phi}{\sqrt{\phi}} e^{-\frac{1}{2} \sqrt{1+6 \log \phi}}. \quad (4.28)$$

However, in a very wide range of the values of argument,  $10^2 \lesssim \phi \lesssim 10^{50}$ , function (4.27) is excellently interpolated by a simple power law (see figure 4)

$$N_B(\phi) \approx 9 \phi^{-5/9}. \quad (4.29)$$



**Figure 4.** The function  $N_B(\phi)$  in (4.27) with  $Z(x)$  given by (4.18) (solid blue line) and its interpolation by (4.29) (dashed pink line) in different ranges of argument. The curves on the right graph practically coincide.

With  $\phi$  being asymptotically given by [see (4.14) and (4.1)]

$$\phi = \frac{2\alpha^2 M_* \Delta\mu_{\text{in}}^2}{\pi^2 \sigma_c T}, \quad (4.30)$$

equations (4.26) and (4.27), (4.29) give the asymptotic behavior of the quantities  $\Delta\mu$  and  $r_B$  as functions of temperature and of their initial values:

$$\frac{\Delta\mu}{\Delta\mu_{\text{in}}} = \left[ \frac{\pi^2 \sigma_c T}{2\alpha^2 M_* \Delta\mu_{\text{in}}^2} \log \frac{2\alpha^2 M_* \Delta\mu_{\text{in}}^2}{\pi^2 \sigma_c T} \right]^{1/2}, \quad \frac{r_B}{r_B^{\text{in}}} = 9 \left( \frac{\pi^2 \sigma_c T}{2\alpha^2 M_* \Delta\mu_{\text{in}}^2} \right)^{5/9} \quad (4.31)$$

## 5 Decay of chiral asymmetry caused by chirality flipping

In the preceding analysis, we totally neglected the flipping term with coefficient  $\Gamma_f(\eta)$  in (2.11), which is justified at high temperatures. However, as the temperature drops down because of cosmological expansion, at some point this term starts dominating over the other terms on the right-hand side of (2.11), after which the chiral chemical potential decays exponentially as

$$\Delta\mu(T) \propto \exp \left( -\frac{\alpha^2 M_* m_e^2}{27 T^3} \right). \quad (5.1)$$

In this section, we estimate the temperature  $T_f$  at which this decay commences.

The contribution to the coefficient  $\Gamma_f(\tau)$  comes from weak and electromagnetic processes, so that we have  $\Gamma_f = \Gamma_w + \Gamma_e$ , where the weak and electromagnetic contributions are estimated, respectively, as (see [18])

$$\Gamma_w \sim G_F^2 T^4 \left( \frac{m_e}{3T} \right)^2, \quad \Gamma_e \sim \alpha^2 \left( \frac{m_e}{3T} \right)^2. \quad (5.2)$$

Here,  $G_F$  is the Fermi constant,  $\alpha$  is the fine structure constant, and  $m_e$  is the electron mass. The factors in the brackets in (5.2) with electron mass  $m_e$  describe suppression of chirality-flipping scattering rates with respect to ‘chirality-preserving’ ones. The weak contribution  $\Gamma_w$  dominates at temperatures  $T > T_{\text{eq}} \simeq \sqrt{\alpha/G_F} \approx 25$  GeV, while, at  $T < T_{\text{eq}}$ , chirality flipping is dominated by the electromagnetic processes and is characterized by  $\Gamma_e$  in (5.2).

In view of equation (3.2), the first term on the right-hand side of (2.11) is itself a sum of two terms with opposite signs:

$$-\frac{c_\Delta \alpha}{\pi^2 \sigma_c} \int_0^\infty \dot{A}(k, \tau) k dk = \frac{2c_\Delta \alpha}{\pi^2 \sigma_c} \int_0^\infty A(k, \tau) k^3 dk - \frac{2c_\Delta \alpha^2 \Delta\mu}{\pi^3 \sigma_c} \int_0^\infty [P(k, \tau) + S_0(k)] k^2 dk. \quad (5.3)$$

In this expression, the thermal ‘tail’ in the spectral function  $A(k, \tau)$  compensates the thermal contribution from  $S_0(k)$ . Indeed, at the ‘tail,’ we have  $A = A_{\text{eq}} = S_0(k_\mu/k)$ , and taking into account (3.9), we observe cancellation of the corresponding integrals in (5.3). Therefore, it is the negative term with the integral of the spectral function  $P(k, \tau)$  in (5.3) that is to be considered. To determine whether the neglect of the flipping term in (2.11) is legitimate, we should compare the absolute value of this term with the absolute value  $\Gamma_f(\tau)\Delta\mu(\tau)$  of the flipping term. It is convenient to divide both quantities by  $\Delta\mu(\tau)$ . For the first expression, we have

$$\frac{2c_\Delta \alpha^2}{\pi^3 \sigma_c} \int_0^\infty P(k, \tau) k^2 dk = \frac{2c_\Delta \alpha^2}{\pi^3 \sigma_c} g_* r_B(\tau) \int_0^\infty S_0(k) k^2 dk = \frac{2\pi c_\Delta \alpha^2}{15 \sigma_c} g_* r_B(\tau), \quad (5.4)$$

Using (4.27) and (4.29) to express  $r_B(\tau)$  through  $r_B^{\text{in}}$ , we estimate (5.4) as

$$\frac{2\pi c_\Delta \alpha^2}{15 \sigma_c} g_* r_B(\tau) \simeq \frac{\pi \alpha^2}{\sigma_c \phi^{5/9}} g_* r_B^{\text{in}}. \quad (5.5)$$

This expression should be compared to each of the quantities in (5.2). Assuming that chirality flipping comes into play at temperatures  $T < T_{\text{eq}} \approx 25$  GeV (this will be confirmed by the final estimate), we only need to take into account the electromagnetic part  $\Gamma_e$ . We then have an equation for the estimate of the temperature of decay caused by chirality flipping:

$$\frac{\pi}{\sigma_c \phi^{5/9}} g_* r_B^{\text{in}} \simeq \left( \frac{m_e}{3T} \right)^2. \quad (5.6)$$

The asymptotic value of  $\phi \gg 1$  is given by (4.30). Substituting it into (5.6) and solving the resulting equation with respect to  $T$ , we obtain

$$T_f = \left( \frac{2^5 \alpha^{10} \sigma_c^4 m_e^{18} M_*^5 \Delta\mu_{\text{in}}^{10}}{3^{18} \pi^{19} [g_* r_B^{\text{in}}]^9} \right)^{1/23} \approx \frac{1.6 \times 10^3}{\sqrt{g_*}} \left( \frac{\Delta\mu_{\text{in}}^{10}}{[r_B^{\text{in}}]^9} \right)^{1/23} \text{ MeV}, \quad (5.7)$$

where we have put the numerical values for physical constants. For  $g_* = 30$ ,  $\Delta\mu_{\text{in}} = 3 \times 10^{-5}$ , and  $r_B^{\text{in}} = 5 \times 10^{-5}$ , this equation gives  $T_f \simeq 150$  MeV (at this time,  $\phi \approx 1200$ ). For  $r_B^{\text{in}} = 5 \times 10^{-4}$ , we obtain  $T_f \simeq 60$  MeV (with  $\phi \approx 3000$ ). This is in good qualitative agreement with the numerical results of [18].

Note that the resulting temperature (5.7) does not depend on the temperature at which the initial values  $\Delta\mu_{\text{in}}$  and  $r_B^{\text{in}}$  are set (and which is assumed to be much higher than  $T_f$ ). This is due to the asymptotic scaling  $\Delta\mu \propto \phi^{-1/2}$  [see (4.26)], and  $r_B \propto \phi^{-5/9}$  [see (4.27) and (4.29)], ensuring that the ratio  $\Delta\mu^{10}/r_B^9$  remains to be roughly constant in the regime of inverse cascade.

## 6 Summary

We provided an analytic treatment of the process of generation of helical magnetic field in an early hot universe in the presence of externally induced lepton chiral asymmetry, and of the

subsequent mutual evolution of the chiral asymmetry and magnetic field. Helical magnetic field is generated from the thermal initial spectrum (extrapolated to all scales including the super-horizon ones) owing to the effects of quantum chiral anomaly. The thermal bath also serves as a medium of relaxation of magnetic field to its thermal state. The generated helical magnetic field and the lepton chiral asymmetry are capable of supporting each other, thus prolonging their existence to cosmological temperatures as low as tens of MeV, with spectral power being permanently transformed from small to large spatial scales (the so-called ‘inverse cascade’) [18].

Our main results are summarized as follows. We obtained analytic expressions describing the evolution of the lepton chiral chemical potential and magnetic-field energy density. For a developed maximally helical magnetic field, both the chiral chemical potential  $\Delta\mu$  and the relative fraction of magnetic-field energy density  $r_B$  depend on their initial values and on time through a single variable  $\phi$  introduced in (4.14). This scaling property is encoded in equations (4.20)–(4.22) and (4.27)–(4.29), and depicted in figures 2 and 4. The late-time asymptotics for  $\Delta\mu$  reproduces the scaling law  $\Delta\mu \propto \eta^{-1/2} \log^{1/2} \eta$  [see equation (4.26)] previously found in this system in [19, 20]. By numerical interpolation, we find that the relative fraction  $r_B$  of the magnetic-field energy density in this regime decays as  $r_B \propto \eta^{-5/9}$  all through the relevant part of the cosmological history. Since the conformal time  $\eta$  in our units is related to the temperature  $T$  as  $\eta = M_*/T$ , this also describes the evolution of these quantities with temperature.

As the temperature drops to sufficiently low values due to the cosmological expansion, the chirality-flipping lepton scattering processes take control over the evolution of chiral asymmetry, leading to its rapid decay (5.1). We derived a simple expression (5.7) for the temperature at which this happens, depending on the initially generated values of the energy density of magnetic field and of the lepton chiral asymmetry.

The analytic expressions obtained in this paper are sufficiently general and may be used for primary evaluation of scenarios of cosmological magnetogenesis by lepton chiral asymmetry.

## Acknowledgments

We are grateful to Alexey Boyarsky and Oleg Ruchayskiy for valuable comments. M. S. and O. T. acknowledge support from the Scientific and Educational Center of the Bogolyubov Institute for Theoretical Physics. The work of O. T. was supported by the WFS National Scholarship Programme and by the Deutsche Forschungsgemeinschaft (DFG) in part through the Collaborative Research Center “The Low-Energy Frontier of the Standard Model” (SFB 1044), in part through the Graduate School “Symmetry Breaking in Fundamental Interactions” (DFG/GRK 1581), and in part through the Cluster of Excellence “Precision Physics, Fundamental Interactions and Structure of Matter” (PRISMA). The work of Y. S. was supported by the Swiss National Science Foundation grant SCOPE IZ7370-152581.

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